Included among these is the sum of the squares of the characteristic numbers of P_1, i.e., the sum of the characteristic numbers of $N_1 = AA^*$; this is the well-known unitary invariant $\sum_{p,q} a_{pq}\bar{a}_{pq}$ of Frobenius.

When A is normal $AA^* = A^*A$ or $P_1U.U^*P_1 = U^*{}_1P.P_1U$ so that $P_1^2 = U^*P_1^2U = (U^*P_1U)^2$. Hence $P_1 = U^*P_1U$ or $UP_1 = P_1U$. Conversely if $UP_1 = P_1U$ we have $A^*A = A^*A$ so that a *matrix* A *is normal when and only when its polar coördinates are commutable, that is, when the polar representations* $A = P_1U$, $A = UP_2$ *coincide.*

It may be mentioned that the above considerations are valid also in the real domain. In this case the polar representation is simply the algebraic formulation of the fact, well known for $n = 3$ from the kinematics of homogeneous linear (non-singular) deformations, that any such deformation may be represented as a superposition of a dilatation and a rotation (the norm AA^* of A determining the ellipsoid of dilatation belonging to the deformation A).

[1] See, for example, Weyl, H.; *Gruppentheorie und Quantenmechanik*, Leipzig, pp. 19–23, 1928.

[2] See Weyl, H., loc. cit.

[3] Since writing the above this problem has been solved and will be treated in a forthcoming note in these PROCEEDINGS.

NOTE ON THE HEAVISIDE EXPANSION FORMULA

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The Expansion Formula solution of the linear differential equation with constant coefficients

$$a_0 \frac{d^n x}{dyn} + a_1 \frac{d^{n-1}x}{dy^{n-1}} + \ldots + a_n y = F$$

was first stated by Oliver Heaviside. Perhaps due to his rather obscure methods of presentation, various writers have stated that the formula was given without proof. Nevertheless, Heaviside gave two proofs of the formula which may be traced through his writings. One of these, which we may designate as the second of Heaviside's proofs, was discussed a few years ago by Vallarta.[1] Heaviside really made no clear point of demarcation between his proofs and in all probability did not believe any proof was necessary. The Expansion Formula was but a single result of his devious analyses in the solution of certain differential equations frequently met in electrical theory. Both proofs hinge on the so-called Conjugate Property of which Heaviside, Rayleigh and Routh made extensive use. Heaviside employed this theorem to great advantage and it serves as a connecting link between the two proofs. The following paragraphs are concerned with the first and restricted proof of the Expansion Formula which up to the present time has received scant attention; later, it will be shown that a proof may be easily and directly established from Rayleigh's[2] investigations of free and forced vibrations.

As has been stated, Heaviside's proof of his Expansion Formula solution is based on the Conjugate Property, which states that the mutual potential and the mutual kinetic energies of two normal systems, r and s are equal at every instant when all the conditions affecting the system are accounted for. Thus symbolically

$$U_{rs} - T_{rs} = 0, \tag{1}$$

where U_{rs} is the mutual potential energy and T_{rs} is the mutual-kinetic energy. Heaviside repeatedly stated the application of this important principle to the solution of linear differential equations. That a proof, constructed along such considerations, should have been so long in forthcoming is indeed rather remarkable. In deriving the Conjugate Property, we shall follow Heaviside's procedure by considering the equations for a line with distributed capacity, inductance and resistance, C, L, and r, respectively, per unit length of line. These may be shown to be

$$\left.\begin{aligned} ri + L\frac{di}{dt} = -\frac{\partial e}{\partial x} &= \frac{1}{C}\frac{\partial^2 i}{\partial x^2} \\ C\frac{de}{dt} &= -\frac{\partial i}{\partial x} \end{aligned}\right\}. \tag{2}$$

From these an equation containing only e may be obtained, but the two above will suffice. The most general solutions are, by the theory of differential equations,

$$e = \sum Au\epsilon^{pt},\ i = \sum Aw\epsilon^{pt} \tag{3}$$

where A is the subsidence constant, of which we shall hear more later, u and w are the normal functions of potential and current (functions of x only), and p is the operator $\frac{d}{dt}$, a root of the determinental equation. The normal solutions which determine the p's are gotten by putting

$$e = u\epsilon^{pt},\ \ i = w\epsilon^{pt} \tag{4}$$

in the first of equations (2) above. The resulting equations will determine the p's and is called the determinental equation.

Thus far, the procedure has been in accordance with the steps usually employed in solving certain forms of differential equations. Heaviside's next step was to obtain an expression for the A's in (3) in terms of the initial terminal conditions, and it was at this point that he threw aside the orthodox methods for solving them and arrived at a general expression by means of the Conjugate Property. This theorem and its applicableness to the solution of the A's in (3) was originally presented in a paper[3] published in 1881, but its complete derivation was first given in an article[4] prepared the following year but published only in his Electrical Papers. Returning to equations (3), we can prove by substitution of equations (4) and algebraical composition that

$$\int_{line}(Cu_1u_2 - Lw_1w_2)dx = \frac{\left[w_1w_2\left(\frac{u_1}{w_1} - \frac{u_2}{w_2}\right)\right]_{line}}{p_1 - p_2} \tag{5}$$

where u_1, u_2 and w_1, w_2 are two admissible values belonging to a pair of normal systems consistent with the p's by putting u, w and p in (4) first equal to u_1, w_1, p_1 and then equal to u_2, w_2, p_2. This is permissible so long as the u's and w's are consistent with the p's. Now, in (5), let p_1 approach p_2; then on going to the limit, $u_1 = u_2$ and $w_1 = w_2$ giving

$$\int_{line}(Cu^2 - Lw^2)dx = \left[w^2 \frac{d}{dp} Z\right]_{line} \tag{6}$$

where Z is the function $\frac{u}{w}$. The interpretations are as follows: Equation (5) gives the excess of mutual potential energy over the mutual kinetic energy of two normal arrangements of potential and current so far as the line is concerned. When account is taken of the terminal conditions, the right-hand members of both (5) and (6) vanish. Equation (6) is a special case of (5) when the normal systems are one and the same. The excess of mutual potential over the mutual kinetic energy is thus given in terms of the potential and current at the ends of the line.

If we concern ourselves with the initial conditions, we have from (3), when $t = 0$

$$\left.\begin{aligned} U &= \sum Au \\ W &= \sum Aw \end{aligned}\right\} \tag{7}$$

U and W are the initial potential and current at one end of the line. Multiplying the first by Cu and the second by Lw, these expressions can be shown to follow

$$\int_{line}(CUu - LWw)dx = A\int_{line}(Cu^2 - Lw^2)dx \tag{8}$$

Thus, utilizing (6)

$$\frac{\int_{line}(CUu - LWw)dx}{\int_{line}(Cu^2 - Lw^2)dx} = \frac{U_{01} - T_{01}}{U_{11} - T_{11}} = \frac{U_{01} - T_{01}}{2(U_1 - T_1)} = \frac{U_{01} - T_{01}}{w^2 \frac{d}{dp} Z} = A \quad (9)$$

where U_{01} and T_{01} are the initial mutual potential and kinetic energies of the initial and normal state as designated by the subscript $_{01}$. By analogy the other terms may be inferred from their proper subscripts. If any energy resides initially in the terminal arrangements, additional terms must be included. As a rule, in most cases met in practice, no energy resides initially at the terminals. Combining (8) and (3), we get

$$i = \sum w \frac{U_{01} - T_{01}}{w^2 \frac{d}{dp} Z} \epsilon^{pt} \quad (10)$$

thus being rid of the A's, which was Heaviside's chief purpose in developing the Conjugate Property. The subsidence solution has now been derived in terms of the normal functions and known conditions. The extension to the derivation of the Expansion Formula was then an easy process. We shall here, however, only briefly state Heaviside's procedure. The reader is referred to either the original discussion[5] employed by Heaviside, or to Vallarta's paper for an excellent treatment.

To obtain the Expansion Formula, having proceeded as far as (10), Heaviside employed an artifice.[5] Taking a circuit with the customary constants, he converts the external forces applied to it into a condenser of infinite capacity, thus creating a subsidence condition. The effect of this substitution is the same as that of a constant applied voltage. By utilizing the conjugate relationship of this system, and substituting in (10) he obtains the final equation of subsidence

$$i = \sum_{j=1}^{j=n} \frac{E\epsilon^{p_j t}}{p_j \frac{d}{dp} Z(p)_{p=p_j}} \quad (11)$$

$$j = 1, 2, 3, \ldots, n.$$

where E is the voltage applied. The complete explicit solution follows thus

$$i = \frac{E}{Z(p)_{p=0}} \sum_{j=1}^{j=n} \frac{E\epsilon^{p_j t}}{p_j \frac{d}{dp} Z(p)_{p=p_j}} \quad (12)$$

$$j = 1, 2, 3, \ldots, n.$$

The first term on the right is the steady state solution. Heaviside at once noted the usefulness of this equation, although the basis for its deduction was by no means general. We shall now turn to a general proof of the formula based on an analysis due to Raleigh[2] with which Heaviside was undoubtedly familiar. The proof is a digression of the alternative Heaviside procedure which has been discussed by Vallarta.[1]

Let $\Psi_1, \Psi_2, \ldots, \Psi_n$ be a set of forces impressed on a system of n-degrees of freedom whose coördinates we may designate as $\psi_1, \psi_2, \ldots, \psi_n$. Write the expressions for the potential, dissipative and kinetic energies in terms of these coördinates, thus

$$\left.\begin{aligned} T &= \tfrac{1}{2} A_{11}\dot{\psi}_1^2 + \tfrac{1}{2} A_{22}\dot{\psi}_2^2 + \ldots + A_{12}\dot{\psi}_1\dot{\psi}_2 + \ldots \\ Q &= \tfrac{1}{2} B_{11}\dot{\psi}_1^2 + \tfrac{1}{2} B_{22}\dot{\psi}_2^2 + \ldots + B_{12}\dot{\psi}_1\dot{\psi}_2 + \ldots \\ U &= \tfrac{1}{2} C_{11}\psi_1^2 + \tfrac{1}{2} C_{22}\psi_2^2 + \ldots + C_{12}\psi_1\psi_2 + \ldots \end{aligned}\right\} \tag{13}$$

$$\dot{\psi} = \frac{\partial\psi}{\partial t}$$

T, Q and U are the kinetic energy, dissipative energy and potential energy, respectively, and A, B and C are constants. Substitute these in the Lagrange-Rayleigh equation

$$\frac{d}{dt}\left(\frac{\partial^2 T}{\partial\dot{\psi}^2}\right) + \frac{\partial Q}{\partial\dot{\psi}} + \frac{\partial U}{\partial\psi} = \Psi. \tag{14}$$

We then obtain the following set of equations

$$\left.\begin{aligned} a_{11}\psi_1 + a_{12}\psi_2 + \ldots &= \Psi_1 \\ \ldots\ldots\ldots\ldots & \\ a_{n1}\psi_1 + a_{n2}\psi_2 + \ldots &= \Psi_n \end{aligned}\right\} \tag{15}$$

where, for compactness, the general constant a_{rs} is the operator

$$A_{rs}\frac{\partial^2}{\partial t^2} + B_{rs}\frac{\partial}{\partial t} + C_{rs}.$$

The solution of any one of the variables is then possible by eliminating all the others, provided the constants and the impressed forces are known. In general

$$\left.\begin{aligned} \nabla\psi_1 &= \frac{\partial\nabla}{\partial a_{11}}\Psi_1 + \frac{\partial\nabla}{\partial a_{12}}\Psi_2 + \ldots \\ \ldots\ldots&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \nabla\psi_n &= \frac{\partial a}{\psi\nabla_{n1}}\psi, + \frac{\psi\nabla}{\partial a_{n2}}\Psi_2 + \ldots \end{aligned}\right\} \tag{16}$$

where Δ defines the determinant

$$\nabla \equiv \begin{vmatrix} a_{11} & a_{12} & \ldots \\ a_{21} & a_{22} & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots \end{vmatrix}. \tag{17}$$

Without any loss of generality, we may put all the forces but one equal to zero. Then we have

$$\nabla \psi_1 = \frac{\partial \nabla}{\partial a_{11}} \Psi_1, \tag{18}$$

or, to avoid subscripts, and putting Z for a_{11}

$$\nabla \psi = \frac{d\nabla}{dZ} \Psi. \tag{19}$$

Now, if all the impressed forces were zero, the determinant (17) above would vanish. That is $\nabla = 0$. If in equations (15), we attempt to obtain the normal solutions by putting $\psi = w\epsilon^{pt}$ after the impressed forces are put equal to zero in them, we obtain the determinental equations of the p's; thus

$$\nabla(p) = 0. \tag{20}$$

Further, suppose that

$$\Psi = E\epsilon^{pt} \tag{21}$$

which is permissible and substitute in (19) after the p's have been written into ∇

$$\psi = \frac{1}{\nabla(p)} \frac{d\nabla(p)}{dZ} E\epsilon^{pt}. \tag{22}$$

If next we assume $\nabla(p)$ is complex, being made up of two components, one real and one imaginary, we may write

$$\nabla(p) = \nabla_1(p) + jp\nabla_2(p), \quad j = \sqrt{-1} \tag{23}$$

∇_1 and ∇_2 are functions of ∇. The first necessarily vanishes, since the period of the force is identical to that obtained from the determinental equation. Therefore,

$$\left.\begin{aligned} \nabla(p) &= jp\nabla_2(p) \\ \frac{d\nabla(p)}{dZ} &= j\nabla_2(p)\frac{dp}{dZ} + jp\frac{d\nabla_2(p)}{dZ} \end{aligned}\right\}. \tag{24}$$

The last term on the right of the second equation is also zero. Substituting these in (22) we get

$$\psi = \frac{1}{jp\nabla_2(p)} j\nabla_2(p) \frac{dp}{dZ} E\epsilon^{pt} \tag{25'}$$

or if i is written for ψ

$$i = \sum \frac{E\epsilon^{pt}}{p\frac{dZ}{dp}}. \tag{25}$$

The $\sum$ denotes summation over the roots of the determinental equation. Equation (25) is identical to the equation of subsidence obtained in (11) according to Heaviside's procedure. It has been deduced, however, without the application of the Conjugate Theorem, although the same limitations hold, namely, that there are no null or repeated roots of p.

[1] M. S. Vallarts, "Heaviside's Proof of His Expansion Theorem," *Jl. A. I. E. E*, April, 1926.

[2] Lord Rayleigh, "Scientific Papers," **1**, 176–187; and "Theory of Sound," 103–142.

[3] Oliver Heaviside, "Electrical Papers," **1**, "On Induction between Parallel Wires," 127–129.

[4] *Ibid.*, "Contributions to the Theory of the Propagation of Current in Wires," 142–148.

[5] Loc. cit., 3, **2**, 372–374.

THE DISTRIBUTION OF CHI-SQUARE

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R. A. Fisher[1] gives a table of χ^2 and states that for large values of n, the number of degrees of freedom in the distribution,

$$\sqrt{2\chi^2} - \sqrt{2n-1} \text{ is normally distributed with } \sigma = 1. \tag{1}$$

It is interesting to ask what other formulas of a similar sort might be used.

When the integrand $f(x)$ of a definite integral vanishes at the limits and has a single maximum, a useful approximation to the value of the integral can sometimes be found by expanding $\log f(x)$ about its maximum $x = m$, writing

$$\varphi(x) = \log f(x) = \varphi(m) + \phi'(x-m) + {}^1/{}_2\phi''(m)(x-m)^2 + \ldots$$

$$\int_a^b f(x)dx = \int_a^b e^{\varphi(m)} e^{\frac{1}{2}\varphi''(m)(x-m)^2}\, dx \text{ (approx.)}$$

$$\text{or } \int_a^b f(x)dx = e^{\varphi(m)} \frac{\sqrt{2\pi}}{\sqrt{-\varphi''(m)}}$$

$$\text{or } \log \int_b^b f(x)dx = \varphi(m) + \frac{1}{2}\log 2\pi - \frac{1}{2}\log\left[-\varphi''(m)\right]$$